# Theory of Exhaust-Plume/Boundary-Layer Interactions at Supersonic Speeds

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**At supersonic speeds, an under-expanded rocket exhaust jet produces a deflection of the external flow and generates a rise in pressure that communicates upstream through the surface boundary layer. For sufficiently large pressure ratios, the viscous layer separates some distance forward of the base and may seriously affect vehicle stability. The present investigation develops an integral method for predicting some of the main features of the near-field interaction between the exhaust plume and the surface boundary layer. The analysis is not restricted to laminar flow and extends to turbulent interactions by utilizing a simplified eddy-diffusivity model. The method also applies to the interaction that occurs in the wake of inclined or asymmetric bodies, and results for the flow over a flat plate at angle of attack are given.**

## Nomenclature

$a$ = velocity profile parameter, Eq. (8); also sonic velocity
$C$ = $(\mu/\mu_\infty)/(T/T_\infty)$
$D$ = $\dfrac{2}{U_e^3}\displaystyle\int_0^\infty \frac{\rho_e}{\rho_\infty}\varepsilon_i\left(\frac{\partial U}{\partial Y}\right)^2 dY$, turbulent dissipation integral
$f$ = $\{2+[(\gamma+1)/(\gamma-1)]\,m_e/(1+m_e)\}\mathscr{H}+(3\gamma-1)/(\gamma-1)+\{(M_e^2-1)/[m_e(1+m_e)]\}Z$
$F$ = $\mathscr{H}+(1+m_e)/m_e$
$\mathscr{H}$ = $\theta_i/\delta_i^*$
$\mathscr{H}_\psi$ = $\dfrac{1}{\delta_i^*}\displaystyle\int_0^{Y_{\psi=0}}\frac{U}{U_e}\left(1-\frac{U}{U_e}\right)dY$
$J$ = $\theta_i^*/\delta_i^*$
$K$ = $\delta_{i_2}^*/\delta_{i_1}^*$
$K_\theta$ = reciprocal of turbulent Reynolds number; $K_\theta = 0.03$
$m$ = $[(\gamma-1)/2]M^2$
$\dot{m}$ = $\displaystyle\int_0^\delta \rho u\, dy$, boundary-layer mass flow
$M$ = Mach number
$R$ = $\dfrac{2\delta_i^*}{U_e^2}\displaystyle\int_0^\infty\left(\frac{\partial U}{\partial Y}\right)^2 dY$, laminar dissipation integral
$Re_x$ = $a_\infty M_\infty x/\nu_\infty$, Reynolds number
$Re_{\delta_i^*}$ = $a_\infty M_\infty \delta_{i_1}^*/\nu_\infty$, Reynolds number
$S$ = streamwise coordinate
$T$ = static temperature
$u, v$ = physical velocity components
$u^*$ = $(u/u_e)_{\psi=0}$, dividing streamline velocity ratio
$U_0$ = $(U/U_e)_{Y=0}$, centerline velocity ratio
$x_0$ = location of beginning of interaction
$y^*$ = distance from centerline to dividing streamline
$Y$ = transformed normal coordinate
$Z$ = $\dfrac{1}{\delta_i^*}\displaystyle\int_0^{\delta_i}\frac{U}{U_e}dY$, velocity integral
$\alpha$ = $u_1/u_2$, velocity ratio
$\beta_1$ = $a_1\rho_1/a_\infty\rho_\infty$
$\beta_2$ = $Ca_1 p_1 M_\infty/a_\infty p_\infty M_1$
$\delta$ = viscous-layer thickness
$\delta_i$ = transformed thickness
$\delta_\psi$ = $(Y/\delta_i^*)_{\psi=0}$
$\delta_i^*$ = $\displaystyle\int_0^\infty\left(1-\frac{U}{U_e}\right)dY$, transformed displacement thickness
$\varepsilon$ = turbulent eddy diffusivity, $\tau=\rho\varepsilon\,\partial u/\partial y$
$\theta$ = $\displaystyle\int_0^\infty\frac{\rho u}{\rho_e u_e}\left(1-\frac{u}{u_e}\right)dy$, momentum thickness
$\theta_i$ = $\displaystyle\int_0^\infty\frac{U}{U_e}\left(1-\frac{U}{U_e}\right)dY$, transformed momentum thickness
$\theta_i^*$ = $\displaystyle\int_0^\infty\frac{U}{U_e}\left(1-\frac{U^2}{U_e^2}\right)dY$, transformed mechanical-energy thickness
$\Theta$ = $\tan^{-1}v_e/u_e$, local streamline inclination
$\lambda$ = $\rho_0 v_0/\rho_1 u_1$, mass-flow parameter
$\mu$ = viscosity coefficient
$\nu$ = Prandtl-Meyer angle; also $\mu/\rho$, kinematic viscosity
$\rho$ = gas density
$\phi$ = zero-shear line inclination

*Subscripts*
cr = critical point
$e$ = local external, inviscid
$i$ = transformed, incompressible
$j$ = jet, exhaust
0 = trailing-edge boundary layer
1 = upper flow
2 = lower flow
$\infty$ – = freestream
$\psi = 0$ = dividing streamline

## I. Introduction

THIS investigation treats the strong interaction between an under-expanded rocket exhaust plume and the boundary layer along the surface of a vehicle. A schematic representation of this type of interaction for laminar flow appears in Fig. 1. Jet expansion at the nozzle exit produces a deflection of the external flow and generates a rise in pressure partially communicated upstream through the viscous layer. For sufficiently large jet-to-freestream pressure ratios, the boundary layer separates forward of the base. An extensive region of separated flow can occur, particularly for laminar boundary layers, and the stability and control of the vehicle may be seriously affected (see Ref. 1, for example). In addition to the evident engineering importance of this problem, it is also an interesting example of the type of fluid-fluid interaction that occurs in the wake of inclined or asymmetric bodies.

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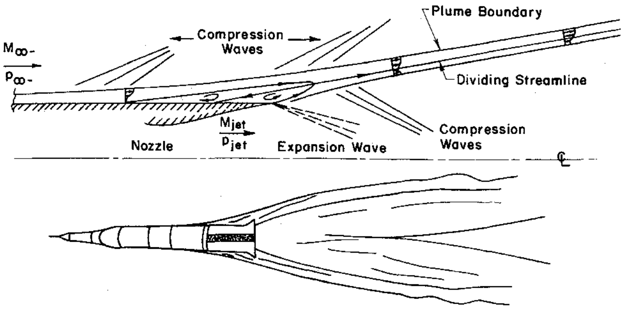


**Fig. 1 Schematic of laminar plume interaction.**

This is a typical strong-interaction problem where the location of the separation point depends on the initial jet deflection which in turn depends on the extent of the region of separated flow. Because of this complexity, the analysis uses an integral or moment method similar to the approach developed by Lees and Reeves[2] to describe the interaction near the vehicle surface. The viscous layer at the edge of the plume is divided into two parts and the boundary-layer equations are integrated across each part and transformed into an equivalent incompressible form. Profile quantities are related to a single parameter through the use of similarity solutions and the flowfield is completely described in terms of six quantities. The distribution of these variables is determined by the solution of a system of nonlinear ordinary differential equations (Sec. II) subject to the appropriate boundary conditions.

To obtain a complete near-field solution for exhaust-plume/boundary-layer interactions, the plume must join the separated-flow region on the body. The necessary joining relations are given by a geometric matching condition and by the requirement for continuity of the edge Mach number, the dividing-streamline velocity ratio, and the boundary-layer mass flow. As originally shown by Crocco and Lees,[3] an integral formulation leads to a subcritical near-wake flow that must pass through a critical point downstream of the region of reversed flow. The plume mixing layer, similar to an asymmetric wake, exhibits this same type of behavior. Initial conditions for any given problem are therefore uniquely determined by the requirement for the correct trajectory to pass through the critical point.

In most interactions, the freestream Mach number and the jet exhaust conditions are specified, requiring an iterative procedure to obtain the solution. The location of the beginning of the interaction is chosen upstream of the trailing edge and, for laminar flow, a departure integral leaving the eigensolution for undisturbed flow toward separation is obtained (see Ref. 4). For turbulent flow, the interaction is initiated through a supercritical-subcritical jump by applying the jump relations derived in Ref. 5 to the turbulent flat-plate boundary layer. The turbulent flow is assumed to be wakelike downstream of the jump, and the surface shear in this region neglected. The equations are integrated through separation to the trailing edge, initial conditions for the plume computation are determined, and the integration is continued in the downstream direction. The critical point is a saddle-type singularity, making it difficult to obtain a numerical solution passing continuously from the subscritical to the supercritical region. As described in Sec. III, the easiest procedure is to integrate away from the critical point to join the upstream branch of the solution.

The results of computations for laminar, transitional and turbulent exhaust-plume/boundary-layer interactions are presented. The analysis can be applied to the interaction occurring in the wake of inclined or asymmetric bodies, and results for the flow over a flat plate at angle of attack are shown.

## II. General Theory for Laminar Flow

### A. Plume Mixing Region

In the solution of many problems involving strong viscous-inviscid interaction, an integral method will simplify the analysis. In the present formulation, the flow in the vicinity of the vehicle is assumed to be adiabatic and locally two-dimensional. The governing system of equations consists of the conservation equations for mass and momentum, and the equation for the rate of change of mechanical energy, obtained by multiplying the momentum equation by the velocity $u$. This additional relation is employed to avoid the use of semiempirical quantities such as the precise entrainment rates required in the Crocco-Lees[3] mixing theory. For nonadiabatic flows, the energy conservation equation can also be included in the basic set of equations.

The viscous layer at the plume boundary is divided into two regions, as shown in Fig. 2, with the $y = 0$ axis defined as the locus of the points of zero shear. The boundary-layer approximations are assumed valid, and the equations are integrated across each portion of the viscous layer and transformed into an equivalent incompressible form using the Stewartson[6] transformation. The resulting system of six first-order, nonlinear ordinary differential equations can be written as follows:

*Equations for upper flow*

$$F_1\, d\delta_{i_1}^*/dS + \delta_{i_1}^*\, d\mathscr{H}_1/dS + f_1\, \delta_{i_1}^*\, d\ln M_1/dS + (\beta_1/m_1)\lambda + (\beta_1/m_1)\phi = (\beta_1/m_1)\Theta_1 \quad (1)$$

$$\mathscr{H}_1\, d\delta_{i_1}^*/dS + \delta_{i_1}^*\, d\mathscr{H}_1/dS + (2\mathscr{H}_1 + 1)\delta_{i_1}^*\, d\ln M_1/dS - \beta_1(1 - U_{0_1})\lambda = 0 \quad (2)$$

$$J_1\, d\delta_{i_1}^*/dS + \delta_{i_1}^*\, dJ_1/dS + 3J_1\, \delta_{i_1}^*\, d\ln M_1/dS - \beta_1(1 - U_{0_1}^{\,2})\lambda = \beta_2 R_1/Re_{\delta_i}^* \quad (3)$$

*Equations for lower flow*

$$F_2\, d\delta_{i_1}^*/dS + \delta_{i_1}^*\, d\mathscr{H}_2/dS + f_2\, \alpha^2\, \delta_{i_1}^*\, d\ln M_1/dS + F_2\, \delta_{i_1}^*\, d\ln K/dS - (\beta_1\, \alpha/m_2\, K)\lambda - (\beta_1\, \alpha^2/m_1\, K)\phi = -(\beta_1\, \alpha^2/m_1\, K)\Theta_2 \quad (4)$$

$$\mathscr{H}_2\, d\delta_{i_1}^*/dS + \delta_{i_1}^*\, d\mathscr{H}_2/dS + (2\mathscr{H}_2 + 1)\alpha^2\, \delta_{i_1}^*\, d\ln M_1/dS + \mathscr{H}_2\, \delta_{i_1}^*\, d\ln K/dS + \beta_1(\alpha/K)(1 - U_{0_2})\lambda = 0 \quad (5)$$

$$J_2\, d\delta_{i_1}^*/dS + \delta_{i_1}^*\, dJ_2/dS + 3J_2\, \alpha^2\, \delta_{i_1}^*\, d\ln M_1/dS + J_2\, \delta_{i_1}^*\, d\ln K/dS + \beta_1(\alpha/K)(1 - U_{0_2}^{\,2})\lambda = \beta_2\, \alpha R_2/K^2 Re_{\delta_i}^* \quad (6)$$

In these equations, the growth of the viscous layer with respect to the $y = 0$ axis is assumed small, and, for example

$$v_1/u_1 = \tan(\Theta_1 - \phi) \approx \Theta_1 - \phi$$

The viscous and inviscid flows are coupled through the Prandtl-Meyer relations for $\Theta_1$ and $\Theta_2$, the induced inclinations of the inviscid streamlines at the edge of the shear layer

$$\Theta_1 = \nu(M_{\infty-}) - \nu(M_1); \quad \Theta_2 = \Theta_j + \nu(M_2) - \nu(M_j) \quad (7)$$

where $\Theta_j$ is the initial angle between the exhaust jet and the freestream.

The shear stresses are zero along the centerline and at the two edges, but mass flow from one region to the other is allowed and $\lambda \neq 0$. The requirement for the pressure and total enthalpy to be constant across the viscous layer is included in the relation for the Mach number derivative

$$d\ln M_2/dS = \alpha^2\, d\ln M_1/dS$$

where

$$\alpha^2 \equiv (u_1/u_2)^2 = (m_1/m_2)(1 + m_2)/(1 + m_1)$$

This equation can be integrated to give the requirement

$$1 + m_2 = C_1(1 + m_1)$$

where $C_1$ is evaluated in terms of the given freestream and jet exhaust conditions, i.e.,

$$C_1 = [(1 + m_j)/(1 + m_\infty)](p_j/p_\infty)^{(\gamma-1)/\gamma}$$

With these expressions, the lower-flow Mach number $M_2(S)$ and the velocity ratio $\alpha(S)$ are functions only of $M_1(S)$, the Mach number at the outer edge.

The plume mixing layer is divided into two regions because the vertical velocity is not specified along a known axis and

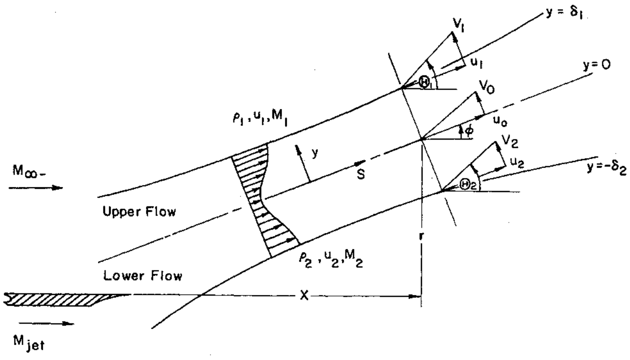


**Fig. 2 Coordinate system for plume mixing layer.**

the physical coordinates of the plume boundary must be obtained as part of the solution. By determining the mass entrained by each portion of the viscous layer, i.e., using both continuity equations, the orientation of the mixing region with respect to the freestream direction can be obtained. The solution of the differential equations thus gives the distribution of $\phi(S)$, which is integrated to obtain the $x$ and $r$ coordinates at any point.

Dividing the plume boundary into two regions requires the introduction of a second length scale, and there are two thicknesses in the problem, $\delta_{i_1}{}^*(S)$ and $\delta_{i_2}{}^*(S)$, or $K(S)$. It is convenient to separate the two flows along the line of zero shear rather than at the dividing streamline, and the mass transfer from one region to the other, $\lambda(S)$, becomes an additional dependent quality. This formulation, however, allows the relations among the different functions of the velocity profiles to be represented by a single profile-shape parameter. The method for obtaining these relations is described in the following sections.

### B. Profile Representation

As previously demonstrated by the present authors and others, appropriate similarity solutions can be used in an integral formulation to determine the interdependence of the different profile quantities. Although no similar solutions exist for a mixing layer in the presence of a pressure gradient, by considering the viscous layer at the edge of the plume to be equivalent to an asymmetric wake, it is possible to use the family of similar wake-like profiles found by Stewartson[7] to describe the velocity distributions. Using these solutions, the various profile quantities can be expressed as polynomial functions of a suitable single-valued profile-shape parameter $a(S)$, (see Appendix), where:

$$a(S) = [f'(\eta)]_{f=0} = (U/U_e)_{\psi=0} \quad \text{for} \quad f'(0) \leq 0$$

and (8)

$$a(S) = [f'(\eta)]_{\eta=0} = (U/U_e)_{Y=0} \quad \text{for} \quad f'(0) \geq 0$$

For use in the integral method, the essential assumption made is that the interdependence of the different profile quantities can be approximated by the equivalent dependence in similar flow, with the relation $\mathscr{H} = \mathscr{H}(a)$, for example, assumed identical in the two cases.

### C. Differential Equations

In the present formulation, the upper-flow shape parameter $a_1(S)$ is taken as one of the dependent variables to be determined by the solution of the differential equations, Eqs. (1–6). The profile parameter for the lower flow $a_2(S)$ relates to $a_1(S)$ by the requirement for a continuous velocity distribution at $y = 0$, the zero-shear line. Thus, for forward $[f'(0) > 0]$ flow, $a_2(S) = \alpha(S)a_1(S)$ whereas, for reversed flow, the relation $U_{0_2}(a_2) = \alpha(S)U_{0_1}[a_1(S)]$ is inverted to give the parameter $a_2(S)$. All profile quantities appearing in the differential equations are written in terms of the dependent profile parameter $a_1(S)$ and the velocity ratio $\alpha(S)$, a function of the upper-flow Mach number $M_1(S)$. To solve the system of equations, however, it is also necessary to express the derivatives of the profile quantities as functions of the derivatives of the dependent variables. For the upper flow, for example

$$d\mathscr{H}_1/dS = \mathscr{H}_{1a}\, da_1/dS$$

where

$$\mathscr{H}_{1a} = \mathscr{H}'(a_1) = (d\mathscr{H}/da)_{a=a_1}$$

The lower-flow profile derivatives are expanded in the following form:

$$d\mathscr{H}_2/dS = \mathscr{H}_{2a}\, da_1/dS + \mathscr{H}_{2m}\, d\ln M_1/dS$$

where

$$\mathscr{H}_{2a} \equiv \partial\mathscr{H}_2/\partial a_1 = \alpha\mathscr{H}'(a_2)[U_{0_1}{}'(a_1)/U_{0_2}{}'(a_2)]$$

and

$$\mathscr{H}_{2m} \equiv \partial\mathscr{H}_2/\partial \ln M_1 = \alpha[(1-m_1/m_2)/(1+m_1)] \times \mathscr{H}'(a_2)[U_{0_1}(a_1)/U_{0_2}{}'(a_2)]$$

Identical relations are obtained for the derivative $dJ_2/dS$. For forward flow, where $U_{0_1}(a_1) = a_1$ and $U_{0_2}(a_2) = a_2$, $U_{0_1}{}'(a_1) = U_{0_2}{}'(a_2) = 1$.

A convenient form for numerical integration is obtained by writing all derivatives in terms of the dependent variables and reducing the number of simultaneous equations by combining the two continuity equations. The remaining system of five first-order differential equations is then inverted to obtain explicit relations for the four unknown first derivatives and for the mass-flow parameter $\lambda(S)$ in the form:

$$(\delta_{i_1}{}^*/M_1)\, dM_1/dS = N_1/D; \quad d\delta_{i_1}{}^*/dS = N_2/D;$$
$$\delta_{i_1}{}^*\, da_1/dS = N_3/D; \quad (\delta_{i_1}{}^*/K)\, dK/dS = N_4/D;$$
$$\lambda = N_5/D \qquad (9)$$

where $D = D[M_1, a_1, K]$; $N_i = N_i[M_1, Re_{\delta_i}{}^*, a_1, K, \Delta\Theta]$; and $\Delta\Theta = \Theta_1 - \Theta_2$.

The angle $\phi(S)$ is obtained from the continuity equation by quadrature:

$$\phi = \Theta_1 - \lambda - (m_1/\beta_1)(f_1 N_1 + F_1 N_2 + \mathscr{H}_{1a} N_3) \qquad (10)$$

and the auxiliary equations

$$dx/dS = \cos\phi; \quad dr/dS = \sin\phi$$

are integrated to find the $x, r$ coordinates of the plume.

### D. Joining Conditions

To obtain a complete near-field solution for exhaust-plume/boundary-layer interactions, the jet plume must join the separated-flow region on the body. In the present formulation, the approximation is made that the boundary-layer interaction upstream of the trailing edge is equivalent to the free interaction that occurs upstream of a compression corner or a forward-facing step. The analysis for the separating boundary-layer

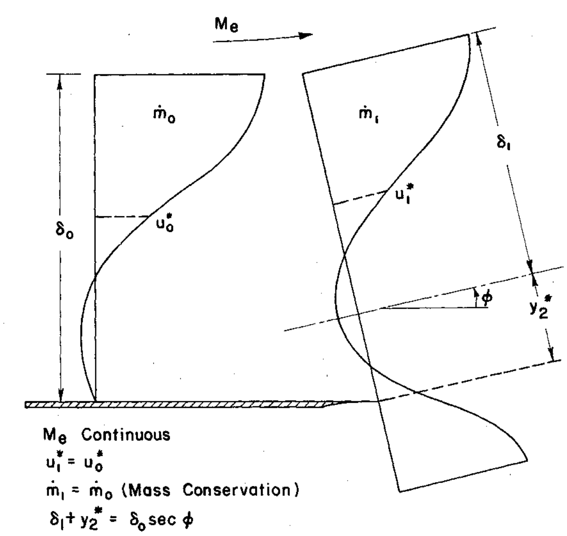


**Fig. 3 Trailing-edge joining conditions.**

flow is therefore identical with that described in Ref. 4. Assuming a location for the beginning of the interaction, a departure integral is obtained leaving the equilibrium solution for undisturbed flat-plate flow toward separation. Integration of the equations proceeds smoothly through the separation point to the specified location of the trailing edge, where initial conditions for the plume computation are determined. Because the initial lower-flow Mach number $M_2$ and the angle $\Delta\Theta$ are functions of $M_1$ for given jet exhaust conditions, there are four unknown initial values: $M_1$, $\delta_{i_1}{}^*$, $a_1$, and $K$, and four joining conditions are necessary to determine them uniquely.

The first three joining conditions (Fig. 3) require continuity of the edge Mach number $M_1$, of the velocity ratio on the upper-flow dividing streamline $u^*$, and of the boundary-layer mass flow $\dot{m}$ which is proportional to $M_1 \delta_{i_1}{}^* Z_1$. The fourth joining condition is a geometric constraint: $(\delta_1 + y_2{}^*)\cos\phi = \delta_0$. In this relation, the dividing streamline for the lower flow is assumed located at the trailing edge, and the height of the viscous layer is required to be continuous. The distance $y_2{}^*$ is given by the expression

$$y_2{}^* = (m_1/\beta_1 \alpha^2)[(1+m_2)/m_2\, \delta_\psi(a_2) + \mathscr{H}_\psi(a_2)]K\delta_{i_1}{}^*$$

and the angle $\phi$ is determined by the continuity equation Eq. (10).

For a given jet Mach number and pressure ratio, initial values for the plume computation are determined by the trailing-edge boundary layer along the upper surface. The initial displacement thickness of the exhaust jet, for example, is not taken into account, although this could be done by modifying the last joining condition. However, the lower viscous layer is always much thinner than the separated upper layer, and as an approximation to the complicated flowfield that exists at the trailing edge, the effect of the nozzle boundary layer on the over-all interaction is neglected. The results obtained for a flat plate at angle of attack, where the boundary layer on the lower surface is relatively thick, appear to justify this assumption (Sec. III. D).

## III. Complete Interactions

### A. Uniqueness Condition

In the solution of exhaust-plume/boundary-layer interactions, the numerators $N_i$ and the determinant $D$ of the system of equations Eq. (9) vanish simultaneously at some point. This location marks the transition from subcritical to supercritical flow,[3,4] and for any given problem, initial conditions are uniquely determined by requiring the correct trajectory to pass through the critical point. The singularity is a saddle point for the system of differential equations, making it difficult to obtain a numerical solution passing continuously from the subcritical to the supercritical region. By determining the eigenvalues and eigenvectors associated with the singularity, however, it is possible to integrate away from the critical point in both upstream and downstream directions. In this manner the complete solution for any interaction can be obtained by matching the integral curve originating at the critical point to the branch of the solution representing the flowfield in the vicinity of the body. This approach is described in the following section.

### B. Method of Solution

In most exhaust-plume/boundary-layer interactions, the free-stream conditions and the jet Mach number and pressure ratio are given in advance. The lower-flow Mach number $M_2$ and the angle $\Delta\Theta$ in Eq. (9) are therefore functions of $M_1$ only, and the critical point is defined by two parameters: $M_{1_{cr}}$ and $K_{cr}$, for example. The remaining variables at the singularity are determined by satisfying the requirement for the numerators $N_i$ and the denominator $D$ to vanish simultaneously.

The complete solution for a specific interaction is obtained by determining the quantities $M_{1_{cr}}$ and $K_{cr}$ such that at any given point $a_1 = a_{1_0}$ the values of $M_1$, $\delta_{i_1}{}^*$ and $K$ on the trajectory passing through the critical point coincide with the values

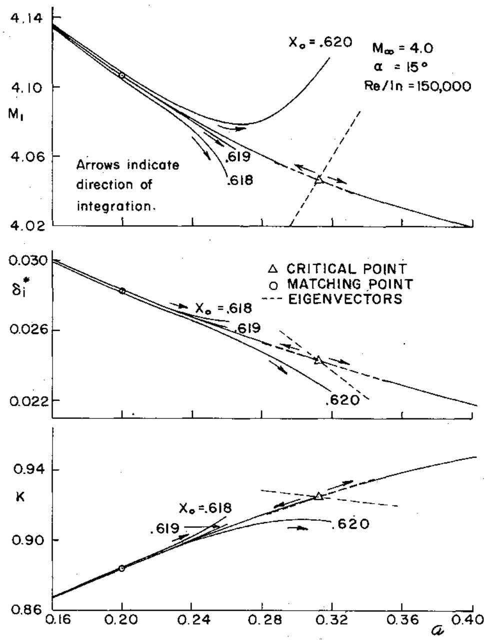


**Fig. 4 Example of method of solution.**

obtained by integrating away from the body. In other words, for the trajectory originating at the singularity, the values of $M_1$, $\delta_{i_1}{}^*$ and $K$ at the point $a_{1_0}$, denoted below by $p_1$, $p_2$, and $p_3$, respectively, can be regarded as functions of the two parameters describing the critical point, $M_{1_{cr}}$ and $K_{cr}$. The values of the same quantities at $a_1 = a_{1_0}$ obtained by integrating away from the body, denoted by $q_1$, $q_2$, and $q_3$, can be regarded as a function of the point $x_0$, the location of the beginning of the interaction. The solution for a particular problem is therefore determined by satisfying the following relations:

$$p_1(M_{1_{cr}}, K_{cr}) - q_1(x_0) = 0; \quad p_2(M_{1_{cr}}, K_{cr}) - q_2(x_0) = 0$$
$$p_3(M_{1_{cr}}, K_{cr}) - q_3(x_0) = 0$$

This set of equations is solved by iterating for both the beginning of the interaction and the corresponding critical-point location. An example of this procedure is shown in Fig. 4, and a more detailed discussion of the method of solution is given in Ref. 8.

### C. Jet Exhaust Interactions

The results of several computations for laminar exhaust-plume/boundary-layer interactions are given in Fig. 5, showing the influence of jet pressure ratio on the plume boundaries and pressure distributions. The under-expanded jet has an effect similar to that of an inclined ramp, although the region of recirculating flow downstream of the trailing edge is less extensive

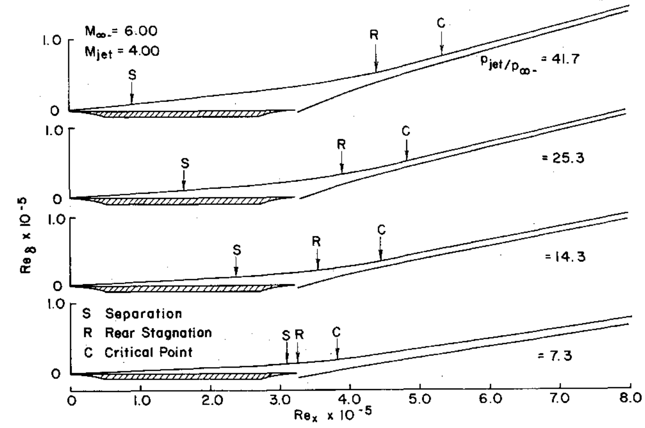


**Fig. 5a Effect of jet pressure ratio on plume boundaries.**

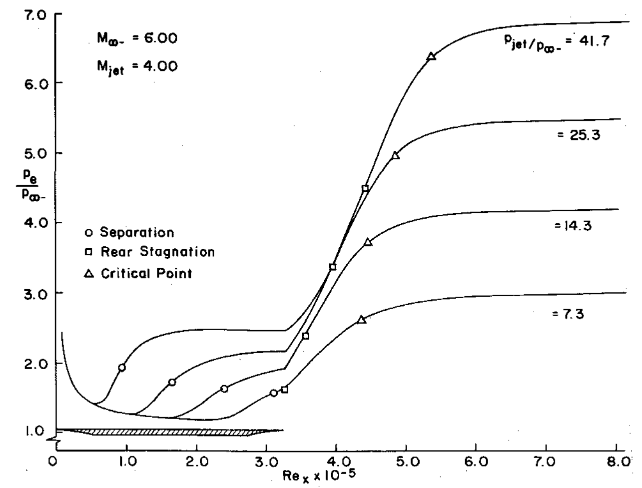


**Fig. 5b Effect of jet pressure ratio on pressure distributions.**

than for an impermeable surface because of the entrainment action of the lower flow. Raising the exhaust pressure increases the plume angle and induces a more extensive region of separated flow, and for the conditions shown, increasing the jet pressure ratio from 7–42 moves the separation point from the trailing edge to a location near the leading edge of the vehicle.

### D. Flat Plate at Angle of Attack

The present method can also be applied to the interaction occurring in the wake of inclined or asymmetric bodies. In particular, the theory for exhaust-plume/boundary-layer interactions can be used without essential modification to study the flow over a flat plate at angle of attack, where the pressure difference is caused by the compression of the lower flow through the leading-edge shock wave and by the expansion of the upper flow. The quantities corresponding to the jet Mach number and pressure ratio are obtained by calculating the boundary layer along the lower surface to determine conditions at the trailing edge. The angle $\Theta_j$ in Eq. (7) is not zero, because of the displacement effect, and is given by the Prandtl-Meyer relation: $\Theta_j = \nu(M_j) - \nu(M_L)$, i.e.,

$$\Theta_2 = \nu(M_2) - \nu(M_L)$$

where $M_L$ is the inviscid Mach number of the lower flow downstream of the leading-edge shock wave. The unit Reynolds numbers for both flows can be determined by assuming a linear temperature-viscosity law, i.e.,

$$Re_{\text{inv}}/Re_\infty = (M_{\text{inv}}/M_\infty)(p_{\text{inv}}/p_\infty)[(1+m_{\text{inv}})/(1+m_\infty)]^{3/2}/C$$

where the subscript inv denotes the inviscid flow downstream of either the expansion fan or shock wave.

The results of computations for an inclined flat plate at

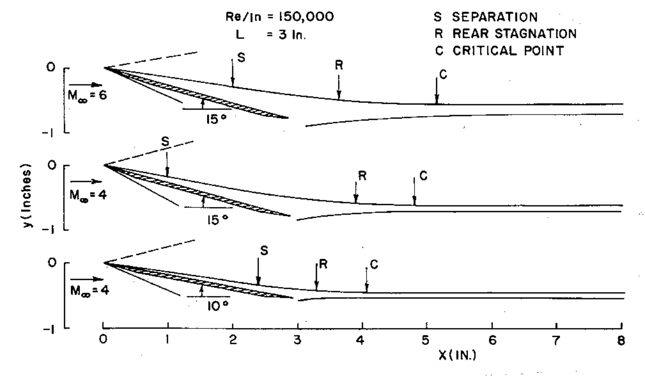


**Fig. 6a Effect of Mach number and angle of attack on wake boundaries.**

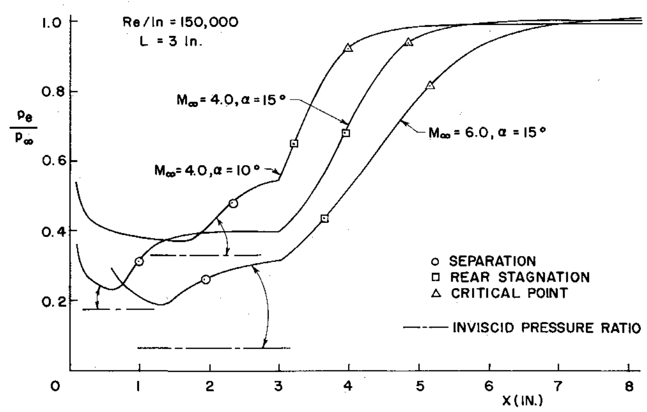


**Fig. 6b Effect of Mach number and angle of attack on pressure distributions.**

$M_\infty = 4.0$ and $M_\infty = 6.0$ are shown in Fig. 6. It is evident that the length of the separated-flow region is extremely sensitive to both the freestream Mach number and to the angle of attack. The distributions of several additional variables in the wake region are shown for one of the configurations in Fig. 7. Most of the effects of angle of attack disappear one plate-length downstream of the trailing edge: the plume is parallel to the freestream direction, the edge Mach number is constant, and the thicknesses of the two flows are equal. The mass transfer $\lambda(S)$ from one region to the other (not shown) is never large ($<10^{-3}$) and vanishes completely, indicating that the zero-shear line and the dividing streamline coincide. The centerline velocity defect decreases slowly, however, and there is an extensive, constant-pressure relaxation region which persists into the far wake.

## IV. Application to Turbulent Flow

The results obtained for laminar exhaust-plume/boundary-layer interactions have shown the integral method capable of predicting the important features of these flowfields. In this section, the extension to turbulent interactions is made by modifying the method developed for laminar flows. The essential aspects of the laminar analysis are retained, and the strong interaction is described by a system of moment integral equations appropriate for turbulent flows. A similar type of integral approach was employed by Alber and Lees[9] for two-dimensional

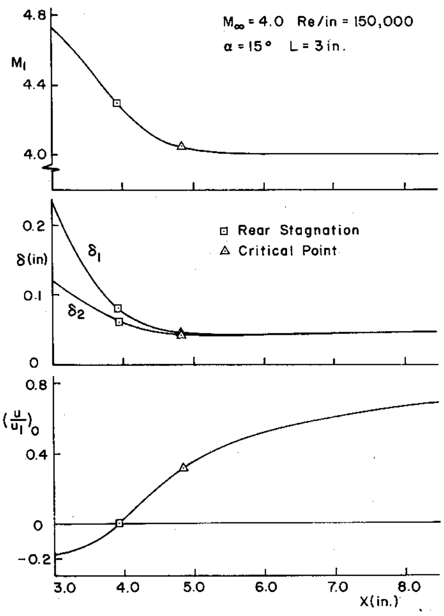


**Fig. 7 Wake Mach number, thickness and centerline velocity distributions.**

wakes, by Fernandez and Lees[10] for flows with large surface injection, and by Todisco and Reeves[11] and Hunter and Reeves[12] for separating and reattaching boundary layers.

### A. Turbulent Separating Flows

As in the case of compressible laminar flows, it is convenient to remove the explicit Mach-number dependence of the basic equations of motion by transforming them into an equivalent incompressible form. Assuming an eddy-viscosity model for the turbulent shear stress

$$\tau = -\overline{\rho u'v'} = \rho\varepsilon\, \partial u/\partial y$$

the modified Stewartson transformation introduced by Alber[9] for wake flows can be used, i.e.,

$$dY = (a_e/a_\infty)(\rho/\rho_\infty)\, dy, \quad dX = (a_e/a_\infty)(\rho_e/\rho_\infty)\, dx$$

provided the eddy viscosity transforms in the following manner:

$$\rho^2 \varepsilon = \rho_R{}^2 \varepsilon_i$$

In the present method, the assumption is made that $\varepsilon_i = K_\theta u_e \theta$, and, as suggested by the experiments of Fernandez and Zukoski,[13] the reference density $\rho_R$ is taken to be $\rho_e$, the edge density.

Near the separation point and in the region of reversed flow, the contribution of molecular viscosity to the shear stress is not important and the friction at the surface can be neglected. With this approximation, the transformed integral equations for the separating turbulent boundary-layer flow can be written in the following form:

$$F\, d\delta_i^*/dx + \delta_i^*\, d\mathscr{H}/dx + f\,\delta_i^*\, d\ln M_1/dx = (\beta_1/m_1)\tan\Theta \tag{11a}$$

$$\mathscr{H}\, d\delta_i^*/dx + \delta_i^*\, d\mathscr{H}/dx + (2\mathscr{H}+1)\,\delta_i^*\, d\ln M_1/dx = 0 \tag{11b}$$

$$J\, d\delta_i^*/dx + \delta_i^*\, dJ/dx + 3J\,\delta_i^*\, d\ln M_1/dx = \beta_1 D \tag{11c}$$

where $D$ is the normalized dissipation integral. Assuming that the turbulent eddy diffusivity is constant across the layer, $D = K_\theta \mathscr{H} R$.

In analogy with the procedure used for laminar flows, the velocity integrals are represented as functions of a single profile-shape parameter, here taken to be the inverse form factor $\mathscr{H} = \theta_i/\delta_i^*$. The required relations $J = J(\mathscr{H})$ and $Z = Z(\mathscr{H})$ are determined from the laminar similarity solutions as $\mathscr{H} \to 0$ (highly separated flow), and from a power-law velocity profile of the form $U/U_e = (Y/\delta)^{1/n}$ as $\mathscr{H} \to 1$. A smooth curve is faired between the two limits by employing the analytical and experimental values compiled by Gran[14] in the intermediate region. To represent the dissipation-function $D(\mathscr{H})$, the integral properties determined by Chapkis[15] for constant-pressure turbulent flows with mass injection are used (see Lees and Chapkis).[16] These solutions are joined smoothly to the reversed-flow values by utilizing the approximation for $D(\mathscr{H})$ given above with $K_\theta = 0.03$.

With the functional relations determined in the manner described, the turbulent flat-plate boundary layer is supercritical and does not respond to downstream disturbances. This property of turbulent flows was established by Crocco,[17] who showed that almost all attached turbulent boundary layers are supercritical. As a result, a strong interaction is initiated by a rapid transition from a supercritical to a subcritical state at some point upstream of the disturbance. In an integral method, as discussed for highly cooled laminar flows in Ref. 4, this sudden change in flow quantities can be approximated by a discontinuity, allowing jumps in the fluxes of mass, momentum and mechanical energy.

The required jump relations were derived in Ref. 5 and can be applied to the turbulent boundary-layer equations without modification. Some typical jumps from $\mathscr{H} = 0.7$ and $\mathscr{H} = 0.8$ are indicated in Fig. 8. Also shown is the boundary between subcritical and supercritical flows, the locus of points at which the determinant of the coefficients in Eq. (11) vanishes. The profile-shape parameter $\mathscr{H}$ is always less than the critical value downstream of the jump, indicating that the boundary layer is subcritical in this region. This subcritical flow interacts with the

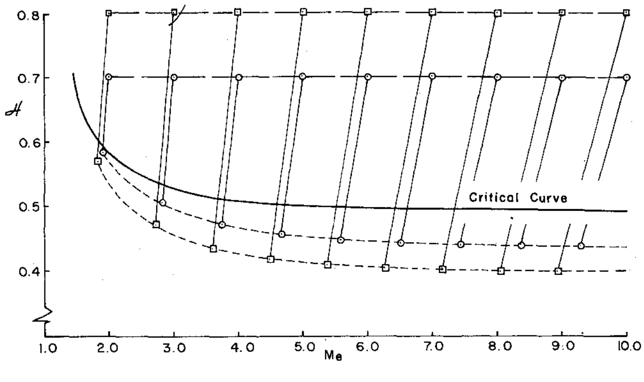


**Fig. 8 Critical curve and jumps for turbulent flow.**

external stream and passes smoothly through separation into the region of reversed flow.

For a given compressible-flow problem, approximate initial conditions immediately upstream of the jump can be obtained from a correlation of flat-plate data or from a transformation theory. In the present investigation, the local friction coefficient $C_f$ is transformed into an equivalent incompressible value $\bar{C}_f$ by means of the direct transformation suggested by Coles.[18] The initial profile-shape parameter $\mathscr{H}$ is then determined as a function of $\bar{C}_f$ by using the experimental results compiled for low-speed equilibrium turbulent boundary layers. Applying the jump relations to this flat-plate flow, conditions downstream of the jump are determined and the differential equations [Eq. (11)] integrated through the separation point. The correct jump location for a particular configuration is determined by the requirement for the boundary layer to return to a supercritical state at some point downstream of the reversed-flow region.

### B. Turbulent Exhaust-Plume Interactions

In the analysis of turbulent plumes, the eddy viscosity is assumed constant across each portion of the mixing layer:

$$\varepsilon_i = K_\theta u_1 \theta_1; \quad 0 \leqq y \leqq \delta_1$$
$$= K_\theta u_2 \theta_2; \quad -\delta_2 \leqq y \leqq 0$$

Although the viscosity is discontinuous at $y = 0$, the shear stress itself remains continuous and vanishes, by definition, along the

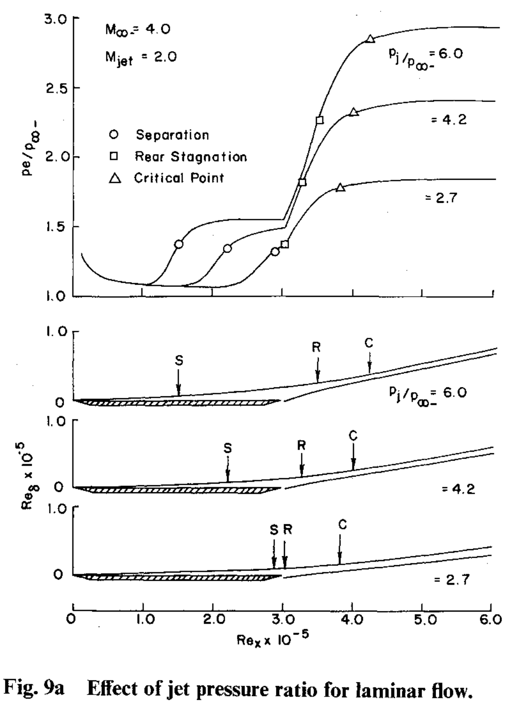


**Fig. 9a Effect of jet pressure ratio for laminar flow.**

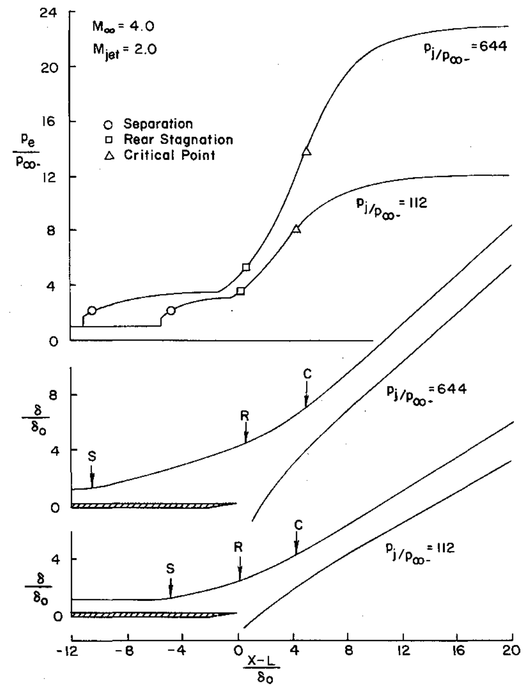


Fig. 9b Effect of jet pressure ratio for turbulent flow.

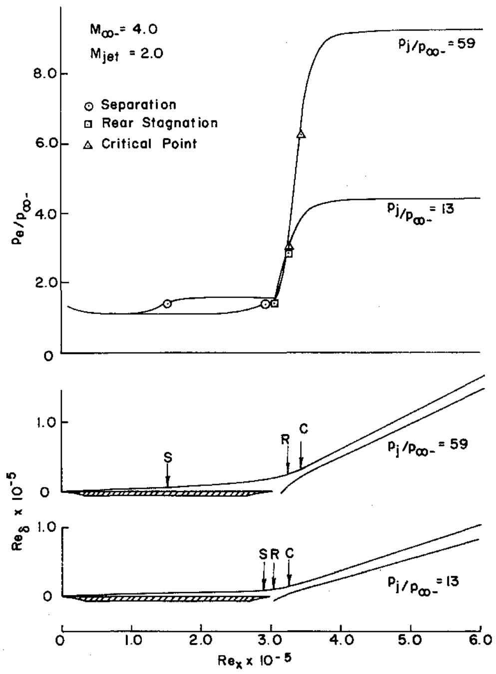


Fig. 9c Effect of jet pressure ratio for transitional flow.

centerline. Following the approach used by Alber and Lees for turbulent wakes, the profile quantities in this region are obtained from the family of similar solutions described in Sec. II. B. The system of equations used for the mixing region at the edge of a laminar plume [Eqs. (1–6)], is therefore applied to turbulent flow by replacing the dissipation functions on the right sides of Eqs. (3) and (6) by $\beta_1 K_\theta \mathscr{H}_1 R_1$ and $\beta_1(m_2 \alpha^2/m_1 K)K_\theta \mathscr{H}_2 R_2$, respectively, with $K_\theta = 0.03$.

Complete solutions for turbulent interactions are obtained by a method similar to the one used for laminar flows. The required procedure is to determine the correct jump location for the upstream branch of the solution such that this integral curve can be continued smoothly through the critical point. The analysis of the singularity is less complicated than for laminar flow because $Re_{\delta_i}{}^*$ does not appear explicitly in the differential equations, and the critical point is defined by a single parameter.

The results of several computations for exhaust-plume/boundary-layer interactions in laminar and turbulent flows are given in Fig. 9. A "transitional" solution, where a laminar boundary layer joins a turbulent plume, is also shown. From a comparison of these figures, it is evident that the overall interaction depends strongly on the nature of the flow, and, in particular, that the jet pressure ratio required to induce boundary-layer separation increases rapidly as the flow changes from a laminar to a fully-turbulent state.

## V. Appendix: Curve-Fits of the Profile Quantities

### A. Reversed Flow Region

$$\mathscr{H} = 0.24710 - 0.43642a - 0.04773a^2 - 0.19654a^3 + 0.41918a^4$$

$$J = 0.37368 - 0.59326a - 0.00220a^2 + 0.27426a^3 - 0.39421a^4$$

$$Z = 1.03285 - 1.35026a + 0.05127a^2 + 0.84914a^3 - 1.89847a^4$$

$$R = 1.25775 + 2.02922a + 3.88529a^2 - 9.20873a^3 + 16.34366a^4$$

$$U_0 = -0.50982a + 0.11855a^2 \qquad a \leqq 0.3$$

$$= 0.05862 - 0.92170a + 0.84007a^2 \qquad 0.3 \leqq a$$

$$\delta_\psi = 1.65927a^{1/2} + 0.16080a - 1.26588a^2 \qquad a \leqq 0.1$$

$$= 0.24115 + 3.04525a - 5.37708a^2 + 4.06049a^3 \qquad 0.1 \leqq a$$

$$\mathscr{H}_\psi = -0.00214a - 0.02542a^2 - 0.64701a^3 + 0.84918a^4$$

where

$$a = [f'(\eta)]_{f=0}; \quad 0 \leqq a \leqq 0.46$$

To evaluate the velocity integral $Z$, the edge of the viscous layer $\eta_e$ was selected at the point where $f'(\eta_e) = 0.99$. Although the Chapman[19] solution corresponds to $a = 0.587$, the centerline velocity ratio $U_0$ begins to increase beyond $a = 0.46$, and, because $U_0$ is necessary to relate the upper and lower flows in the present method, this value represents the limiting reversed-flow profile.

### B. Forward Flow Region

$$\tilde{\delta} = 2.35860 - 5.27389a + 6.73235a^2 - 6.12945a^3 + 2.34196a^4$$

$$\tilde{\mathscr{H}} = 0.58530 + 0.59289a - 2.14390a^2 + 1.48259a^3 - 0.52313a^4$$

$$\tilde{J} = 0.88674 + 0.67519a - 1.66397a^2 + 0.28925a^3 - 0.19051a^4$$

$$\tilde{Z} = 2.43208 + 0.98139a - 0.91905a^2 + 1.01789a^3 - 0.52020a^4$$

$$\tilde{R} = 0.53431 - 0.58670a - 0.62596a^2 + 0.89792a^3 - 0.22179a^4$$

where

$$a = [f'(\eta)]_{\eta=0} = U_0; \quad 0 \leqq a \leqq 1$$

Because the velocity-defect integral

$$\tilde{\delta} = \int_0^\infty (1 - f')\, d\eta$$

vanishes as $a \to 1$, it was more convenient to curve-fit the similar profile functions before normalization. The required quantities are given by the relations: $\mathscr{H} = \tilde{\mathscr{H}}/\tilde{\delta}$, $J = \tilde{J}/\tilde{\delta}$, $Z = \tilde{Z}/\tilde{\delta}$, $R = \tilde{R}\tilde{\delta}$. And, for example

$$d\mathscr{H}/da = (\tilde{\delta}\, d\tilde{\mathscr{H}}/da - \tilde{\mathscr{H}}\, d\tilde{\delta}/da)/\tilde{\delta}^2$$

For the integral $Z$ in this region, the edge of the viscous layer was chosen at the point where the velocity defect has decreased to 1% of its centerline value, i.e., $[1 - f'(\eta_e)] = 0.01[1 - f'(0)]$.

### C. Joining Region

At the joining point, it is necessary to determine the dividing-streamline velocity ratio for the separated boundary layer at the trailing edge. The required curve-fit was obtained from the family of similar boundary-layer solutions

$$u_{BL}{}^* = 0.05496a_0 + 0.29702a_0{}^2 + 11.19943a_0{}^3 - 21.66525a_0{}^4 + 10.06854a_0{}^5$$

where $a_0$ is the boundary-layer profile shape parameter (see Ref. 4).

### D. Turbulent Flow

$$J = 0.04326 + 1.32866\mathscr{H} - 0.84434\mathscr{H}^2 + 4.52910\mathscr{H}^3 - 3.97206\mathscr{H}^4 \qquad \mathscr{H} \leqq 0.6$$

$$= 4\mathscr{H}/(3-\mathscr{H}) \qquad 0.6 \leqq \mathscr{H}$$

$$Z = 0.26382 + 1.58825\mathscr{H} + 11.59151\mathscr{H}^2 - 26.01280\mathscr{H}^3 + 27.21594\mathscr{H}^4 \qquad \mathscr{H} \leqq 0.7$$

$$= 2\mathscr{H}/(1-\mathscr{H}) \qquad 0.7 \leqq \mathscr{H}$$

$$D = 0.11921\mathscr{H} - 0.50858\mathscr{H}^2 + 0.78287\mathscr{H}^3 - 0.41607\mathscr{H}^4$$

The assumptions required to obtain these profile relations are discussed in Sec. IV. A.

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